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On Instability in Geometric Evolution Equations

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ABSTRACT. A general parabolic evolution equation is considered for a closed hypersurface in Euclidean space. All stationary solutions are shown to be Lyapunov unstable if the normal velocity of a hypersurface depends only on its normal and second fundamental form and is independent of its position. Instability of time periodic solution is also discussed.

1. Introduction

This is a preliminary note. We consider the initial value problem of an evolution of a closed hypersurface Γ_t in $\mathbb{R}^n$

$$V = f(\mathbf{n}, -\mathbf{A}). \tag{1}$$

Here $\mathbf{n}$ is an inward unit normal vector field on Γ_t and V is normal velocity in the direction of $\mathbf{n}$; $\mathbf{A} = -\,\mathrm{d}\,\mathbf{n}$ denotes the second fundamental form. We shall prove that all stationary solutions S of (1) is Lyapunov unstable provided that (1) is (nondegenerate) parabolic. This generalized a recent work of Ei and Yanagida [EY] where they assumed that f defends on $\mathbf{A}$ only through its mean curvature. Their method is completely different from ours. They linearized equation around stationary solution and appeal to spectral analysis. Their method applies to the equation depending on space variable but invariant under translation. We simply use a distance function of S and appeals to the maximum principle. We believe our proof is simpler than theirs for this problem.

Our method also applies to instability of the periodic solution of

$$V = f(t, \mathbf{n}, -\mathbf{A}). \tag{2}$$

where f is time periodic. We show that periodic solutions S_t are unstable unless second fundamental form vanishes somewhere on S_t for all t.

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2. Parabolic evolution equations

We formulate our equations as in [GG1]. Let E be a bundle over the sphere S^{n-1} of the form

$$E = \{(\overline{p}, Q_{\overline{p}}(X)) \in S^{n-1} \times \mathbb{S}_n; X \in \mathbb{S}_n\}$$

with $Q_{\overline{p}}(X) = R_{\overline{p}} X R_{\overline{p}}$ and $R_{\overline{p}} = I - \overline{p} \otimes \overline{p}$; $R_{\overline{p}}$ is the projection orthogonal to $\overline{p}$. Here $\mathbb{S}_n$ denotes the space of the $n \times n$ real symmetric matrices. By the standard Euclidean metric the bundle E is identified with the tensor bundle $\mathrm{T}\,S^{n-1} \otimes \mathrm{T}^*\,S^{n-1}$ over S^{n-1}. Let f be a function from $[0, \infty) \times E$ to $\mathbb{R}$. We shall always assume that f is at least continuous. Suppose that a hypersurface Γ is given as a zero level set of u in $\mathbb{R}^n$ such that the gradient $\nabla u \neq 0$ on Γ and $\mathbf{n} = \nabla u / |\nabla u|$. Then as in [GG1], the second fundamental form (in the direction of $\mathbf{n}$) is of the form

$$\mathbf{A} = -Q_{\overline{p}}(\nabla^2 u)/|\nabla u| \quad \text{with } \overline{p} = \nabla u / |\nabla u|, \tag{3}$$

where $\nabla^2 u$ denotes the Hessian of u in space variables.

We recall the notion of parabolicity of the equation

$$V = f(t, \mathbf{n}, -\mathbf{A}) \tag{4}$$

for evolving hypersurface Γ_t. It is convenient to introduce the level set equation

$$u_t + F_f(t, \nabla u, \nabla^2 u) = 0 \tag{5}$$

$$\text{with} \qquad F_f(t, p, X) = |p| f(t, p/|p|, Q_{\overline{p}}(X)/|p|). \tag{6}$$

This equation is uniquely determined if each level set of u moves by (4) and a super level set $u > c$ is "inside" the level set $u = c$. Sign in (6) are different from those in [CGG] because our convention of $\mathbf{n}$ is opposite.

We say (4) is *strictly parabolic* (uniformly in t) if for each $M > 0$ there is $\mu > 0$ such that

$$F_f(t, \bar{p}, X + Y) - F_f(t, \bar{p}, X) \leq -\mu \operatorname{trace}(Q_{\bar{p}}(Y)) \tag{7}$$

for all $Y \geq 0$, $|X| \leq M$, $|\bar{p}| = 1$, $t \in [0, \infty)$, where $|X|$ is the operator norm of X as a selfadjoint operator. If (7) holds for $\mu = 0$ we say (4) is (degenerate) *parabolic.* A level set method [CGG], [ES] provides a unique global generalized solution. The following version is taken from [GG1].

2.1. Unique global existence. *Suppose that (4) is parabolic. Let Γ_0 be the boundary of a bounded open set in $\mathbb{R}^n$. Then there is a unique generalized solution $\{\Gamma_t\}_{t\geq 0}$ of (4) starting from Γ_0.*

If f and Γ_0 are smooth enough and (4) is parabolic, there is a local-in -time classical solution Σ_t (see e.g. [GG2]). Moreover Σ_t agrees with Γ_t as far as the former exists [GG2] (see also [ES] for the mean curvature flow). So our generalized solution is a natural extention of classical solution.

3. Instability of stationary solution

We say that C^2 hypersurface S is *stationary* for

$$V = f(\mathbf{n}, -\mathbf{A}) \tag{8}$$

if $f(\mathbf{n}, -\mathbf{A}) = 0$ on S. Let $U(\alpha)$ denote a tubular neighborhood of S of the form

$$U(\alpha) = \{\, x \in \mathbb{R}^n;\ \mathrm{dist}(x, S) < \alpha\},$$

where dist denotes the distance. We say that S is *Lyapunov stable* for (8) if for each $\epsilon > 0$ there is $\delta > 0$ such that a (generalized) solution Γ_t with initial data Γ_0 stays in $U(\delta)$ for all $t > 0$ provided that Γ_0 is contained in $U(\epsilon)$. If not, S is called *unstable.* If α is a supremum of α' such that every point of $U(\alpha')$ has a unique nearest point on S, α is called the *reach* of S and dented by reach S.

3.1. Instability Theorem.*Suppose that (8) is strictly parabolic. Let S be a stationary C^2 hypersurface of (8) such that $S = \partial D$ for some open set D in $\mathbb{R}^n$. Suppose that* reach $S = \alpha_0 > 0$, $\inf_S |\mathbf{A}| = \sigma > 0$ *and that* $|\mathbf{A}|$ *is bounded on* S *(if* S *is not compact). For* $0 < \alpha < \alpha_0$ *let*

$$\Gamma^\alpha = \{x \in D; \mathrm{dist}(x, S) = \alpha\}. \tag{9}$$

Let Γ_t^α *be a (generalized) solution of (8) starting from* Γ^α. *Then there are* α_1 *(* $0 < \alpha_1 < \alpha_0$ *) and* $c_0 = c_0(f, \sigma, S) > 0$ *such that*

$$\mathrm{dist}(\Gamma_t^\alpha, S) \geq (\alpha + c_0 t) \wedge \alpha_1 \quad \textit{for all } t > 0 \tag{10}$$

where $a \wedge b = \min(a, b)$. *The same inequality holds if* D *in (9) is replaced by its complement.*

3.2. Corollary. *Suppose that* S *is a stationary* C^2 *closed hypersurface for (8) with non-vanishing second fundamental form. Then* S *is (Lypunov) unstable provided that (8) is strictly parabolic.*

Of course Lyapunov instability follows from (10).

3.3. Remark on noncompact surface. Following formula for distance function is a key for the proof of Theorem 3.1. Let v be the signed distance function of S, *i.e.*,

$$v(x) = \begin{cases} \mathrm{dist}(x, S) & \text{for } x \in D \\ -\mathrm{dist}(x, S) & \text{otherwise.} \end{cases}$$

If S is C^2, then so is d which is proved in [GT; §14].

Even for a non bounded open set generalized solution can be constructed by levelset method; see Ilmanen [I] and Ishiiand Souganidis [IS].

3.4. Lemma. *For a general C^2 hypersurface $S = \partial D$ with an open set D*

$$\nabla^2 v(y)(I - v(y)\nabla^2 v(y))^{-1} = \nabla^2 v(x), \quad y = x + v\mathbf{n},\ x \in S$$

for $|v| <$ reach S.

This is also a key in [GG2], where the local existence of classical solution is proved for (2) by a level set method.

3.5. Proof of Theorem 3.1. We set

$$w(t,x) = v(x) - \rho(t) \quad \text{with } \rho(t) = \alpha + c_0 t.$$

Our goal is to take $c_0 > 0$ so that w is a supersolution of the level set equation of (8):

$$u_t + F(\nabla u, \nabla^2 u) = 0, \qquad F = F_f$$

in a set $U_+(\alpha_1) \setminus U_+(\alpha/2)$, where

$$U_+(\alpha) = \{x \in D; \mathrm{dist}(x, S) < \alpha\}.$$

If such a c_0 exists, comparison principle ([CGG], [GG1]) implies that Γ_t^α is contained in $\{w \geq 0\}$. This yields (10).

Since S is stationary, we see

$$F(\nabla v, \nabla^2 v) = 0 \quad \text{on } S.$$

By Lemma 3.4 this yields

$$F(\nabla v, \nabla^2 v(I - v\nabla^2 v)^{-1}) = 0 \quad \text{in } U(\alpha_0).$$

This implies

$$w_t + F(\nabla w, \nabla^2 w) = -c_0 + F(\bar{p}, X) - F(\bar{p}, X(I - vX)^{-1}) \quad \text{in } U_+(\alpha_0). \tag{11}$$

with $\bar{p} = \nabla v$, $X = \nabla^2 v$.

We take $\alpha_1 > 0$ small so that

$$|\nabla^2 v| \geq \sigma/2 \qquad \text{in } U_+(\alpha_1) \text{ (by } \inf |\mathbf{A}|), \tag{12}$$

$$(I - v\nabla^2 v)^{-1} \geq I/2 \quad \text{in } U_+(\alpha_1). \tag{13}$$

Since $X\bar{p} \otimes \bar{p} = 0$ by $|\nabla v| = 1$, we see

$$Q_{\bar{p}}(X^2(I - rX))^{-1} = X^2(I - rX)^{-1} = -\frac{1}{v}(X - X(I - vX)^{-1})$$

By parabolicity

$$\begin{aligned} F(\bar{p}, X) - F(\bar{p}, X(I - vX)^{-1}) &\geq \mu v \operatorname{trace} Q_{\bar{p}}[X^2(I - rX)^{-1}] \\ &= \mu v \operatorname{trace} X^2(I - rX)^{-1} \quad \text{on } U_+(\alpha_1). \end{aligned}$$

with $M = \sup_{U_+(\alpha_1)} |\nabla^2 v|$. Using (12), (13) we see

$$\operatorname{trace} X^2(I - rX)^{-1} \geq \frac{1}{2}\left(\frac{\sigma}{2}\right)^2 = c_1$$

which yields

$$F(\bar{p}, X) - F(\bar{p}, X(I - vX)^{-1}) \geq \mu c_1 v \quad \text{on } U_+(\alpha_1).$$

If we set $c_0 = \mu c_1 \alpha/2$, from (11) it follows that w is a classical supersolution of the level set equation of (8) in $U_+(\alpha_1) \setminus \overline{U_+(\alpha/2)}$.

The proof for the last statement is parallel so is omitted.

3.6. General instability Theorem. *For (8) there is no stable stationary C^2 closed hypersurface provided that (8) is strictly parabolic and that f is C^1.*

Proof. By Corollary 2.2 we may assume that there is a point on at which

$$(\mathbf{n}, \mathbf{A}) = (\bar{p}_0, O), f(\mathbf{n}, \mathbf{A}) = (\bar{p}, O)$$

for some $\bar{p}_0 \in S^{n-1}$. Since S is stationary,

$$f(\bar{p}_0, O) = 0.$$

The following lemma implies the nonexistence of closed stationary solution, so the proof is complete.

3.7. Nonexistence Lemma. *Suppose that (8) is strictly parabolic and f is C^1. Suppose that $f(\bar{p}_0, O) = O$ for some $\bar{p}_0 \in S^{n-1}$. Then there is no stationary C^2 closed hypersurface for (8)*

Proof. Let S be a stationary C^2 closed hypersurface. Since S is compact, there is a half space H such that

$$H = \{x + c \in \mathbb{R}^n; \quad x \cdot \bar{p}_0 \geq 0\}, \qquad S \subset H \text{ with } c \in S.$$

Note that ∂H is a stationary solution of (8) since $f(\bar{p}_0, O) = 0$ }. Since (8) is strictly parabolic and f is C^1 we may apply the strong maximum principle and conclude S cannot touch ∂H for $t > 0$. This contradicts the existence of stationay closed hypersuface S.

3.8. Remark. If f is C^1 in $\mathbf{A}$, the parabolicty is equivalent to say that $\partial f / \partial \mathbf{A}$ is positive definite.

4. Instability of periodic solutions

We consider

$$V = f(t, \mathbf{n}, -\mathbf{A}), \tag{14}$$

where $f : [0, T] \times E \to \mathbb{R}$ is continuous and T-periodic, *i.e.* $f(t, \bar{p}, -\mathbf{A}) = f(t + T, \bar{p}, -\mathbf{A})$. We say $S_t \, (-\infty < t < \infty)$ is a T-periodic $C^{2,1}$ solution of (14) such that $S_t = S_{t+T}$ where $C^{2,1}$ implies that C^2 in space and C^1 in time. Note that the signed distance function v of S_t is now a $C^{2,1}$ function.

Let $U(\alpha, t)$ denote the α-tubular neighborhood of S_t. We say S_t is *Lyapunov stable* for (14) if for each $\epsilon > 0$ there is $\delta > 0$ such that a generalized solution Γ_t with $\Gamma_t|_{t=t_0} = \Gamma_0$ always stays in $U(\epsilon, t)$ for all $t > t_0$ provided that $\Gamma_{t_0} \subset U(\delta, t_0)$.

In some cases S_t is called S_t is called T-periodic even if S_t is T-periodic up to T-periodic translation $\mathbf{a}(t)$, *i.e.*, $S_{t+T} = S_t + \mathbf{a}(t)$. Here $\mathbf{a}(t)$ is asuumed to have the form

$$\mathbf{a}(t) = \mathbf{b}(t + T) - \mathbf{b}(t)$$

for some C^1 function $\mathbf{b}(t)$. If we set

$$\Sigma_t = S_t - \mathbf{b}(t)$$

then $\Sigma_t = \Sigma_{t+T}$ and Σ_t solves

$$V = f + \mathbf{n} \cdot \mathbf{b}(t).$$

Thus by T-periodic solution we shall always mean T-periodic with no ambiguity of translation.

4.1. Instability Theorem. Suppose that (14) is strictly parabolic. Let S_t be a T-periodic C^2 solution of (8) of closed hypersurfaces surrounding an bounded open set D in $\mathbb{R}^n$. Suppose that $\inf_{S_t} |\mathbf{A}| = \mathbf{a}(t) \not\equiv 0$. Let $\alpha_0 > 0$ denote the minimum of reach S_t in t. For $0 < \alpha < \alpha_0$, $t_0 \in \mathbb{R}$

$$\Gamma^\alpha = \{x \in D\,;\, \mathrm{dist}(x, S_{t_0} = \alpha\} \quad \text{for } t > t_0.$$

Let Γ_t^α be a (generalized) solution of (14) with $\Gamma_t^\alpha = \Gamma^\alpha$ at $t = t_0$. Then there are $\alpha_1(S_t)(0 < \alpha_1 < \alpha_0)$ and nonnegative T-periodic function $c_0(t)(\not\equiv 0)$ depending only on f, $\mathbf{a}$, S_t such that

$$\mathrm{dist}(\Gamma_t^\alpha, S_t) \geq (\alpha + \int_0^t c_0(\tau)\, \mathrm{d}\,\tau) \wedge \alpha_1 \quad \text{for all } t > 0.$$

The same inequality holds if D in the definition of Γ^α replaced by $\mathbb{R}^n \setminus D$.

Proof. As in the proof of Theorem 3.1 we set

$$w(x,t) = v(x,t) - \rho(t), \quad \rho(t) = c_0' \int_0^t a(\tau)^2\, \mathrm{d}\,\tau,$$

and choose $c_0' > 0$ in suitable way so that w is a supersolution of the level set equation of (14). Here v denotes the signed distance function of S_t.

4.2. Corollary. *Suppose that (14) is strictly parabolic. If S_t is a T-periodic $C^{2,1}$ solution of (14) consisting of closed hypersurfaces, then S_t is Lyapunov unstable if* $\inf_{S_t} |\mathbf{A}| \not\equiv 0$ *as a function of time.*

This is an imediate from Theorem 4.1.

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